Suggested solutions of HW4

Ex 16.3

27. Because $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} = 0$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y}$, then F is conservative.

$\frac{\partial f}{\partial x} = \frac{2x}{y}$, so $f(x,y) = \frac{x^2}{y} + g(y)$. And $\frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2}$

So $g'(y) = \frac{1}{y^2}$ $\quad g(y) = -\frac{1}{y} + C$. Hence $f(x,y) = \frac{x^2-1}{y} + C$.

32. Because $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = -2x\sin y = \frac{\partial M}{\partial y}$, F is conservative.

$\frac{\partial f}{\partial x} = 2x\cos y$, so $f(x,y,z) = x^2\cos y + g(y,z)$. And $\frac{\partial f}{\partial y} = -x^2\sin y + \frac{\partial g}{\partial y} = -x^2\sin y$,

so $\frac{\partial g}{\partial y} = 0$. $g(y,z) = h(z)$ $\quad \frac{\partial f}{\partial z} = h'(z) = 0$. So $h(z) = C$

Thus $f(x,y,z) = x^2\cos y + C$

(a) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = [x^2\cos y]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = [x^2\cos y]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = [x^2\cos y]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$.

(d) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = [x^2\cos y]_{(1,0)}^{(1,0)} = 1 - 1 = 0$.

Ex 16.4

20. $M = 4x - 2y$, $N = 2x - 4y$. So $\frac{\partial M}{\partial y} = -2$, $\frac{\partial N}{\partial x} = 2$.

Work $= \oint_C (4x-2y)\,dx + (2x-4y)\,dy = \iint_R [2-(-2)]\,dx\,dy = 4\iint_R dx\,dy$

$= 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$.

22. $M = 3y$, $N = 2x$. So $\frac{\partial M}{\partial y} = 3$, $\frac{\partial N}{\partial x} = 2$

$\oint_C 3y\,dx + 2x\,dy = \iint_R (2-3)\,dx\,dy = \int_0^{\pi}\int_0^{\sin x} (-1)\,dy\,dx = -2$.

26. $M = x = a\cos t$, $N = y = b\sin t$. So $dx = -a\sin t\,dt$, $dy = b\cos t\,dt$

So Area $= \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \pi ab$

28. C_1: $M = x = t$, $N = y = 0$, so $dx = dt$, $dy = 0$.

C_2: $M = x = (2\pi - t) - \sin(2\pi - t) = 2\pi - t + \sin t$. $N = y = 1 - \cos(2\pi - t)$

$dx = (\cos t - 1)dt$, $dy = \sin t\, dt$

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\oint_{C_1} x\,dy - y\,dx + \frac{1}{2}\oint_{C_2} x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} 0\,dt + \frac{1}{2}\int_0^{2\pi} [(2\pi - t + \sin t)(\sin t) - (1 - \cos t)(\cos t - 1)]dt$$

$$= -\frac{1}{2}\int_0^{2\pi} (2\cos t + t\sin t - 2 - 2\pi\sin t)\,dt$$

$$= -\frac{1}{2}[3\sin t - t\cos t - 2t - 2\pi\cos t]_0^{2\pi} = 3\pi$$

34. Let $M = y$, $N = 0$. Then $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 0$. Thus $\oint_C M\,dx + N\,dy = \iint_R (\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x})dy\,dx$

So $\oint_C y\,dx = \iint_R 0 - 1\, dy\,dx$. Thus $-\oint_C y\,dx = \iint_R dx\,dy = \text{Area of } R = \int_a^b f(x)dx$.

35. Let $\delta(x,y) = 1$. Then $\bar{x} = \frac{M_y}{M} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\iint_R x\,dA}{A}$

So $A\bar{x} = \iint_R x\,dA = \iint_R (x + 0)\,dx\,dy = \oint_C \frac{x^2}{2}dy$.

$A\bar{x} = \iint_R x\,dA = \iint_R (0 + x)\,dx\,dy = -\oint_C xy\,dy$

and $A\bar{x} = \iint_R x\,dA = \iint_R (\frac{2}{3}x + \frac{1}{3}x)\,dx\,dy = \oint_C \frac{1}{3}x^2 dy - \frac{1}{3}xy\,dx$

Hence $\frac{1}{2}\oint_C x^2 dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2 dy - xy\,dx = A\bar{x}$.

36. If $\delta(x,y) = 1$, then $I_y = \iint_R x^2\delta(x,y)\,dA = \iint_R x^2\,dA = \iint_R (x^2 + 0)\,dy\,dx = \frac{1}{3}\oint_C x^3 dy$,

$\iint_R x^2\,dA = \iint_R (0 + x^2)\,dy\,dx = -\oint_C x^2 y\,dx$, and $\iint_R x^2\,dA = \iint_R (\frac{3}{4}x^2 + \frac{1}{4}x^2)\,dy\,dx$

$= \oint_C \frac{1}{4}x^3 dy - \frac{1}{4}x^2 y\,dx = \frac{1}{4}\oint_C x^3 dy - x^2 y\,dx$

Thus $\frac{1}{3}\oint_C x^3 dy = -\oint_C x^2 y\,dx = \frac{1}{4}\oint_C x^3 dy - x^2 y\,dx = I_y$.

38. $M = \frac{1}{4}x^2 y + \frac{1}{3}y^3$, $N = x$. So $\frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2$, $\frac{\partial N}{\partial x} = 1$

And $\text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (\frac{1}{4}x^2 + y^2) > 0$ inside $\frac{1}{4}x^2 + y^2 = 1$

Hence work $= \int_C F\cdot dr = \iint_R (1 - \frac{1}{4}x^2 - y^2)\,dx\,dy$ will be maximized on the region $R = \{(x,y) \mid \text{curl}(F) \geq 0\}$ or over the region enclosed by $1 = \frac{1}{4}x^2 + y^2$.